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Dynamical Bifurcations and Competing Instabilities in Landau and Landau–Ginzburg Theory

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We discuss the case of a Landau theory for systems with competing instabilities and the behavior of its solutions at the phase transition. We show that when one takes into account the finite speed of variation of control parameters in any real experiment, the jumping effects characteristic of dynamical bifurcation theory lead to an enhancement of the critical mode; we then apply our discussion to the Landau–Ginzburg equation. We corroborate our discussion by the results of numerical simulations.

1. INTRODUCTION

Dynamical bifurcations were first studied by Neishtadt (1988), who showed how these differ from the standard bifurcation picture; the interest in dynamical bifurcations then diffused among mathematicians and, to a somewhat lesser extent, physicists; a panorama of progress in the theory and applications is offered by the conference proceedings and the bibliography collected in Benoit (1991).

Here we discuss dynamical bifurcations in the framework (and the terminology) of the Landau theory of phase transitions, including the Ginzburg–Landau equation describing the case of a local order parameter (Landau and Lifshitz, 1958). In particular, we are interested in the situation of *competing instabilities*, in which case the effects related to dynamical bifurcation can be expected to play a significant role in enhancing the most unstable mode or, as we will call it, the *critical mode*. We substantiate the heuristic reasoning leading to such an expectation by means of numerical computations, clearly showing the discussed effect.

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This phenomenon is of particular interest in the framework of a Landau–Ginzburg-type equation; indeed, in physical problems described by these, one observes the formation of regular patterns corresponding to critical modes, which are not fully understood by the theory. The main difficulty in analyzing these transitions in terms of bifurcation theory (Guckenheimer and Holmes, 1993) lies indeed in the presence of a continuous spectrum (given in physical terms by a dispersion relation) (Collet and Erkmann, 1990), so that the bifurcation theorems (Guckenheimer and Holmes, 1993), which allow one to reduce the dynamics to unstable modes and ensure the other modes are slaved to these, do not apply here. Nevertheless, consideration of concrete cases shows that—although we have necessarily a whole interval of unstable frequencies, i.e., modes, once the zero solution has lost stability—the observed dynamics is well described in terms of the mode first becoming unstable, i.e., the critical mode alone. Notice that in nondegenerate situations, the critical mode will also be the most unstable one, at least for values of the control parameter near the critical one.

We will discuss how the effect of dynamical bifurcations enhancing the critical mode can be used to understand the above-mentioned behavior, and again will substantiate our discussion by numerical simulations.

As already mentioned, our discussion will be entirely in terms of Landau and Landau–Ginzburg theory; the reader desiring a greater mathematical generality and sophistication in the general discussion of dynamical bifurcations is referred to Benoit (1991).

2. DYNAMICAL BIFURCATIONS

Let us briefly illustrate the main point of dynamical bifurcations, in the frame and with the language of Landau theory; a more complete introduction is provided by Lobry (1991).

Let us consider the case of a scalar order parameter $x \in \mathbf{R}$; we have then to deal with a potential $V(x)$ and, up to rescalings,

$$\dot{x} = -\nabla V(x) \equiv \lambda x - x^3 \tag{1}$$

so that for $\lambda < 0$ we have the stable solution $x = 0$, and for $\lambda > 0$ there are two stable solutions $x = x_\pm = \pm\sqrt{\lambda}$. The prediction (of standard bifurcation theory) is therefore that, if we gradually increase λ starting from a negative value, we observe $x = 0$ for $\lambda < 0$, and $x^2 = \lambda$ for $\lambda > 0$.

Nevertheless, this can happen in no real experiment in which we observe the evolution of x as λ is varied. Indeed, although for $\lambda > 0$ the limit of $x(t)$ under (1) is $x_\pm$, this limit is reached only in an infinite time, and in a real experiment we only have finite time, so that at most we can hope to observe $x(\lambda) \simeq x_\pm$. It was remarked by Neishtadt that the situation is actually much worse: indeed, for $\lambda \simeq 0$, the dynamics of (1) around $x = 0$ undergoes a *critical slowing down* (indeed linear terms vanish), and this implies that the equilibrium solutions $x_\pm$ do not even approximate the effective solutions over any finite time, no matter how long.

A more realistic description of an experimental situation in which λ is varied very slowly but still with finite speed would consider, instead of (1), the system

$$\begin{cases} \dot{x} = \lambda x - x^3 \\ \dot{\lambda} = \varepsilon \end{cases} \tag{2}$$

Now, if at $t = 0$ we have $\lambda = 0$, $x = x_0 \simeq 0$, since the x equation reads now

$$\dot{x} = \varepsilon t x - x^3 \tag{3}$$

then taking the linear approximation for small x, we get

$$x(t) \simeq e^{\varepsilon t^2/2} x_0; \qquad x(\lambda) = e^{\lambda^2/(2\varepsilon)} x_0 \tag{4}$$

This means that $x(t)$ will remain essentially constant up to a certain time t^* [equivalently, $x(\lambda)$ up to a certain value λ^* of the control parameter], after which it undergoes an explosive behavior. Obviously, once x begins to grow, the linear approximation is no longer valid, and we enter the nonlinear regime; the saturation of the nonlinearity ensures that we do not actually have an explosion, but a sudden jump from the regime described by (4) to a regime of saturated nonlinearity, in which $x \simeq x_\pm$. We have therefore

$$x(t) \simeq x_0, \quad t < t^*; \qquad x(t) = x_\pm(\lambda(t)), \quad t > t^* \tag{5$'$}$$

$$x(\lambda) \simeq x_0, \quad \lambda < \lambda^*; \qquad x(\lambda) = x_\pm(\lambda), \quad \lambda > \lambda^* \tag{5$''$}$$

Clearly, this is an idealization, in that the transition between the two regimes is not at a definite value of t, λ, but over a (narrow) jumping interval around t^*, λ^*.

The above qualitative discussion is easily checked to be right by numerically integrating (2) and plotting $x(\lambda)$; see Fig. 1.

A rough estimate for λ is given by asking that $|\dot{x}| \simeq (d/dt)|x_\pm(\lambda)|$, i.e.,

$$2\varepsilon t x \simeq \left(\frac{\varepsilon}{t}\right)^{1/2} \tag{6}$$

Since in the linear regime (4) $x \simeq x_0$, this gives

$$t^* \simeq \left(\frac{1}{2\varepsilon x_0^2}\right)^{1/3}; \qquad \lambda^* \simeq \left(\frac{\varepsilon^2}{2x_0^2}\right)^{1/3} \tag{7}$$

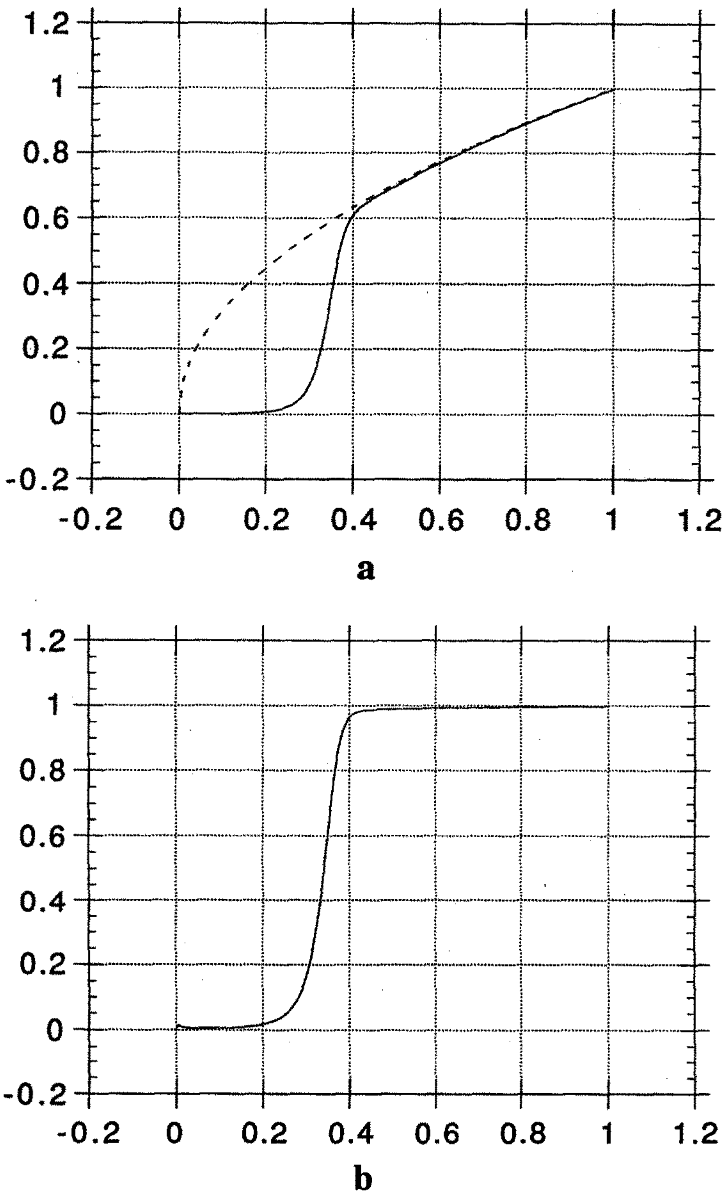

Fig. 1. Dynamical bifurcation versus standard one. (a) The $x(\lambda)$ resulting from numerical integration of (2) (solid line), to be compared with the functions $x_b(\lambda) = \sqrt{\lambda}$ predicted by standard bifurcation theory (dashed line). (b) The ratio $\rho(\lambda) = x(\lambda)/x_b(\lambda)$, showing the abrupt jump from the linear regime to the saturated nonlinear one. We used $\varepsilon = 0.01$, time step $dt = 0.01$, and initial data $\lambda_0 = 0$, $x_0 = 0.001$.

Finally, we remark that (2) is invariant under the rescaling (α a real number)

$$x \to \alpha x; \qquad \lambda \to \alpha^2 \lambda; \qquad t \to \alpha^{-2} t; \qquad \varepsilon \to \alpha^4 \varepsilon \tag{8}$$

The scaling property ensures that Fig. 1 describes the general behavior of (2), and is not only valid for the ε and x_0 used in the concrete numerical integration. It also shows that if ε is reduced, the jumping will take place at the smallest values of λ, i.e., that in the limit $\varepsilon \to 0^+$ we recover the standard bifurcation picture.

3. LANDAU THEORY WITH COMPETING INSTABILITIES

We would now like to consider how the behavior discussed in the previous section affects the situation in which we have competing instabilities. We consider the case of two instabilities, arising at nearly degenerate values of the control parameter, λ_1 and $\lambda_2 = \lambda_1 + \delta\lambda$; the order parameter will have two components x, y corresponding to the amplitudes of these. In other words, we have

$$\begin{cases} \dot{x} = \lambda x - (x^2 + y^2) x \\ \dot{y} = (\lambda - \delta\lambda) y - (x^2 + y^2) y \\ \dot{\lambda} = \varepsilon \end{cases} \tag{9}$$

Notice that the scaling (8) does leave (9) invariant, provided $\delta\lambda$ scales in the same way as λ.

Let us first consider $\varepsilon = 0$ and $\lambda \gg \delta\lambda$, with small initial datum $(x_0, y_0) \simeq (0, 0)$. It is clear that we will first have a regime in which $x(t)$, $y(t)$ grow at nearly the same rate; when $r^2 \equiv x^2(t) + y^2(t)$ becomes of order λ the dynamics enters a slow regime drifting between the circles $r^2 = \lambda - \delta\lambda$ and $r^2 = \lambda$ until reaching the minima of the effective potential $(x, y)_\pm = (\pm\sqrt{\lambda}, 0)$. We have therefore that r and θ (polar coordinates for x, y) evolve on different time scales; this behavior is concretely shown in Fig. 2.

Let us now consider the dynamical bifurcation case, $\varepsilon \neq 0$. The discussion of the previous section shows that (for small initial datum) we expect jumplike behavior for $x(t)$ and $y(t)$; but due to the term $\delta\lambda$, the jump from the linear to the saturated nonlinear regime would now take place at different values of λ for x and y. Taking $\lambda_0 = 0$, from (7) we have

$$\lambda^*_{(x)} \simeq \left(\frac{\varepsilon^2}{2x_0^2}\right)^{1/3}; \qquad \lambda^*_{(y)} \simeq \left(\frac{\varepsilon^2}{2y_0^2}\right)^{1/3} + \delta\lambda \tag{10}$$

Notice that different behaviors are possible:

(i) If $\lambda^*_{(x)} < \lambda^*_{(y)}$, the system will jump directly from the linear regime $(x(\lambda), y(\lambda)) \simeq (x_0, y_0)$ to equilibrium solutions $(x, y) \simeq (x_\pm, 0)$, with $x_\pm = \pm\sqrt{\lambda}$.

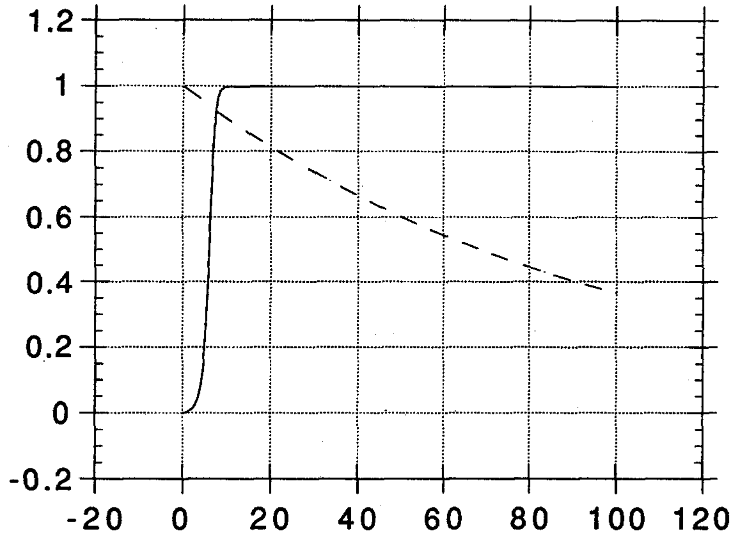

Fig. 2. Numerical integration of (9) for $\varepsilon = 0$. Here we plot the values of $r(t)$ (solid line) and $y(t)/x(t)$ (dashed line). We used $\lambda = 1$, $\delta\lambda = 0.01$, time step $dt = 0.01$, and initial data $x_0 = y_0 = 0.001$.

(ii) If $\lambda^*_{(x)} \simeq \lambda^*_{(y)}$, the system jumps to a state in which $r^2 \simeq \lambda$, $y/x \simeq y_0/x_0$, and then slowly readjusts to reach $(x, y) = (x_\pm, 0)$; this differs from the $\varepsilon = 0$ case in that the r expansion is jumplike.

(iii) If $\lambda^*_{(x)} > \lambda^*_{(y)}$, the system jumps first to $(x, y) \simeq (0, \pm\sqrt{\lambda})$ and then, with a slow dynamics, reaches $(x, y) = (x_\pm, 0)$.

If the initial data are of the kind $x_0 \simeq y_0$, we are in case (i), so that the noncritical mode y is suppressed by the jumping behavior.

Pictorially, the available energy is not given gradually to the system, but absorbed in a big bunch (it can then be redistributed between modes on a much slower time scale); in the transition from the linear to the nonlinear regime, the mode that starts first to absorb energy takes all of it.

It should be stressed that in (i)–(iii) above, one should understand that $\lambda^*_{(x)} \simeq \lambda^*_{(y)}$ actually means that their difference is smaller than the width of the jumping region; a similar criterion applies for the order relations in (i) and (iii).

An explicit numerical integration of (9) is shown in Fig. 3; notice that by the scaling properties of (9) this illustrates the general case.

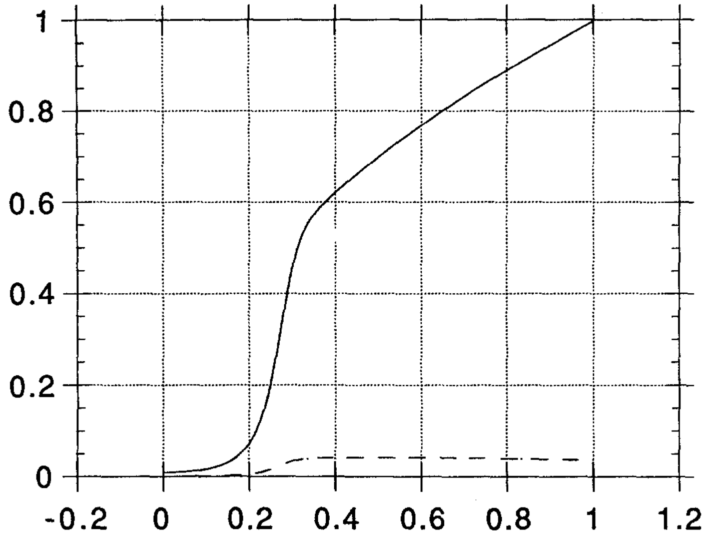

Fig. 3(a)

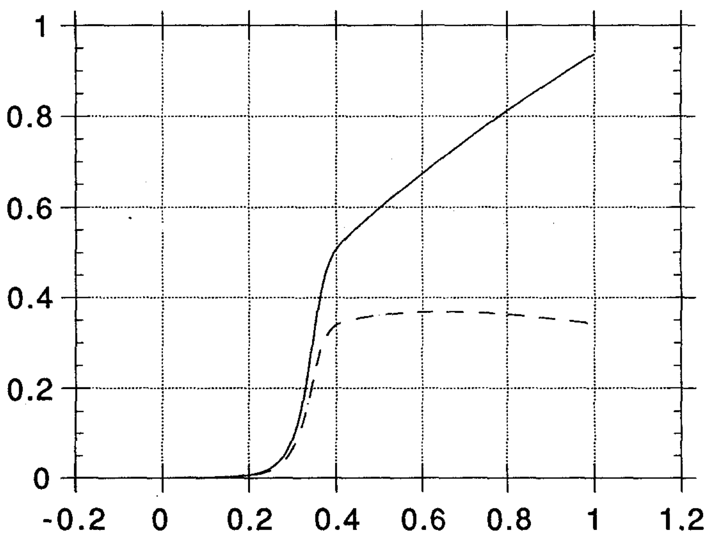

Fig. 3(b)

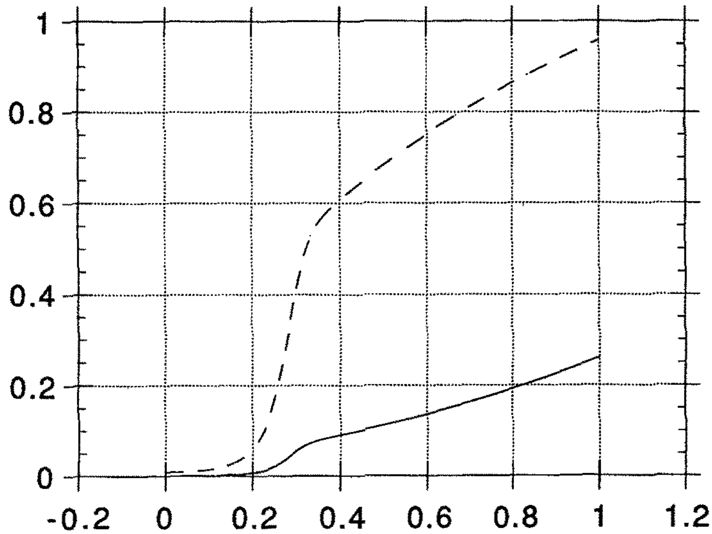

Fig. 3(c)

Fig. 3. Numerical integration of (9) for $\varepsilon \neq 0$ and different initial data. We plot the values of $x(\lambda)$ (solid line) and $y(\lambda)$ (dashed line). We used $\varepsilon = 0.01$, $\delta\lambda = 0.01$, $\lambda_0 = 0$, and time step $dt = 0.01$. (a–c) Cases (i)–(iii), respectively, discussed in Section 4. The initial data used were (a) $x_0 = 0.01$, $y_0 = 0.001$, (b) $x_0 = y_0 = 0.001$, and (c) $x_0 = 0.001$, $y_0 = 0.01$.

4. LANDAU–GINZBURG EQUATION AND DYNAMICAL BIFURCATIONS

Let us now discuss how the previous considerations can be of use in the study of bifurcations for the Landau–Ginzburg (GL) equation (Collet and Eckmann, 1990; Newell *et al.*, 1993; Eckhaus, 1992; Van Harten, 1991)

$$u_t = (\lambda + \Delta)u - |u|^2 u \tag{11}$$

where Δ is the Laplacian. Similar consideration actually apply to any equation of the form

$$u_t = L(\lambda, \Delta)u - |u|^2 u \tag{12}$$

with L a linear operator, such as, e.g., the Swift–Hohenberg equation

(Collet and Eckmann, 1990), obtained for $L(\lambda, \Delta) = [\lambda - (1 + \Delta)^2]$. We will stick to the GL case, both for concreteness and for its physical (Collet and Eckmann, 1990; Newell *et al.*, 1993) and mathematical (Eckhaus, 1992; van Harten, 1991) interest; it will be clear how to generalize our discussion to (12). We will moreover consider $u \in \mathbf{R}$, $x \in \mathbf{R}$, but nothing essential would change if we considered $\mathbf{R}^m$, $\mathbf{R}^n$ instead.

By Fourier transforming $u(x, t)$ so that with standard notation

$$u(x, t) = f_k(t)|k\rangle = \int f_k(t) e^{ikx}\, dk \tag{13}$$

one finds for one-dimensional GL equation (11)

$$\dot{f}_k = [\sigma(\lambda, k) - |f|^2] f_k \equiv (\lambda - k^2) f_k - |f|^2 f_k \tag{14}$$

where obviously $|f|^2 = \int |f|^2\, dk$; notice that the critical mode $k = 0$ will remain the most unstable one. Obviously, if we impose boundary conditions, e.g., $u(\pm a, t) = 0$, only a discrete infinite set of Fourier modes will be present.

If we start from small $u_0 \simeq 0$ and negative λ, gradually increasing the latter, at $\lambda = k_0^2$ all the frequencies $-k_0 \le k \le k_0$ should be excited. On the basis of linear analysis, for small $|f|^2$ we would have

$$\dot{f}_k = \sigma(\lambda, k) f_k \equiv (\lambda - k^2) f_k \tag{15}$$

and considerations similar to those presented in the previous section for $\varepsilon = 0$ would apply.

In the dynamical bifurcations setting, we should supplement (11) and (14) with $\dot{\lambda} = \varepsilon$, $\varepsilon > 0$, and the considerations of the previous section would apply as well. In particular, if the energy is smoothly distributed among modes—so that in a small interval $-k_0 \le k \le k_0$ it is essentially constant—we are in case (i) of the previous section.

Clearly, in the continuous spectrum case we should speak more properly of packets of frequencies rather than of single frequencies, so that the final outcome of our discussion in the GL case is that the (jumping) effects of dynamical bifurcation, for energy initially smoothly distributed, enhances the critical mode and leads to a direct transition to a regime in which only frequencies $k \simeq 0$ have nonvanishing amplitudes.

This heuristic discussion requires a numerical check; we have indeed considered a discretized version of (14) in which only modes $k = nk_0$ are present, $n = 0, \pm 1, \ldots, \pm 16$. The results of numerical integration of this are displayed in Fig. 4; they confirm and substantiate our discussion.

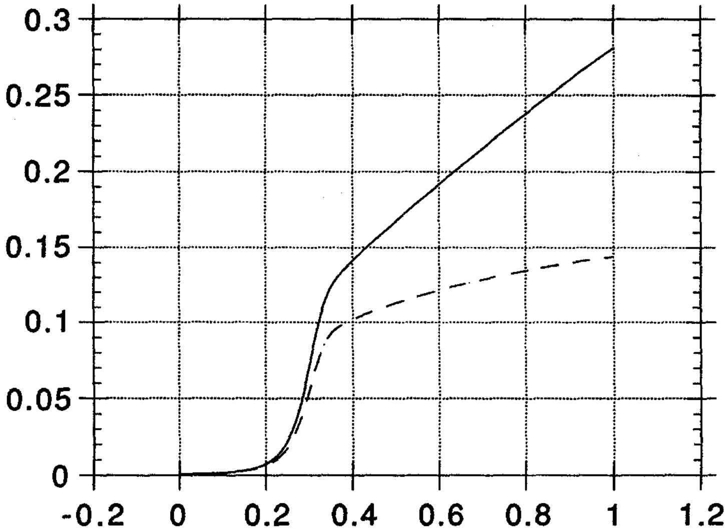

Fig. 4. Numerical integration of (14). We keep modes $k = nk_0$, with $k_0 = 0.01$ and $n = 0, \pm 1, \ldots, \pm 16$. We chose $\varepsilon = 0.01$, $\delta\lambda = 0.01$, $\lambda_0 = 0$; as initial data we gave $f_k(0) = 0.001$. The solid line represents the amplitude f_0 of the critical mode $k = 0$ as a function of λ; the dotted line represents the average amplitude of the other modes, $F = [\sum_{k \neq 0} f_k]/32$.

6. CONCLUSIONS

We have shown that, in the presence of competing instabilities with nearly degenerate instability thresholds and initial energies, taking into account the finite speed of variation of the control parameter leads to an enhancement of the critical modes, i.e., of the modes which first become unstable and remain, at least for a small interval of values of the control parameter, the most unstable ones. This enhancement is due to the essentially discontinuous and "jumplike" behavior of the mode amplitude versus the control parameter, characteristic of dynamical bifurcations.

We have shown how this affects some simple models, and more important, the GL equation, by means of numerical simulations which confirmed and substantiated our discussion.

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